Applications of topology for evaluating pictorial structures

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Abstract

The direct application of methods derived from mathematical topology is outlined to obtain algorithmic *and* parameter-free definitions of image objects. Image objects are the starting point for any further evaluation say extraction of quantitative features. The hit-or-miss transformation well known from mathematical morphology is used to generate images, called *ricefield* images. These images contain, comparable with region growing algorithms, regions which correspond to topological properties like path connectedness and local extrema. Additionally neighborhood relations between such regions can be derived. The latter allows subsequent mathematical morphological processing on an elavated level of abstraction.

Introduction

Pattern recognition and image analysis are impicitly concerned with objects represented by sets of pixels in digitized images. In pattern recognition the final goal is to recognize the object in the image under consideration. Conversely in image analysis the object in an image is primarily mapped to a set of quantitative features. These features or measurements combined with external properties allow subsequent interpretation of the image objects.

In pattern recognition *and* image analysis the most cumbersome processing step is the definition, segmentation or delineation of the actual object inside of the image. Numerous publications exist describing methods and algorithms performing the task [4, 11]. Most of the proposed algorithms contain explicit and implicit parameters adjusted to the problem.

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Using concepts from mathematical topology a method is described applying only procedural declarations but no *a-priori* knowledge like numeric parameters. The declarations are applied to devide an arbitrary digital grey value image into different objects. Additionally these objects are related to each other which is one condition for further evaluation of image content.

The methods used base on minmax operations, mentioned in [4] and more recently by [13]. Relations exist to the concept of medial axis, see [1]. The algorithms used are already published in [5, 8]. In this article the results and the possible applications of the transformations are outlined in more detail.

Applications of topology

Fundamental definitions

The methodology used stems from the fields of algebraic topology and mathematical morphology [8, 9], especially from the theory of homotopy. The latter deals mostly with paths, their properties as well as the possibility to find topologically equivalent classes of paths. In the simplest case, this leads to the genus or the connectivity number which is in digital images an important property. It will be shown, that the connectivity number is a side product of the transformations and operations described. In Fig.1 an object is sketched with two topological not equivalent paths. This is directly related to a class of objects with holes, here with a genus of 0. The genus is defined as the number of holes minus the number of voids.

In the following we will not fulfill all requirements of mathematical rigorousness. Otherwise this would lead us beyond the scope of this article.

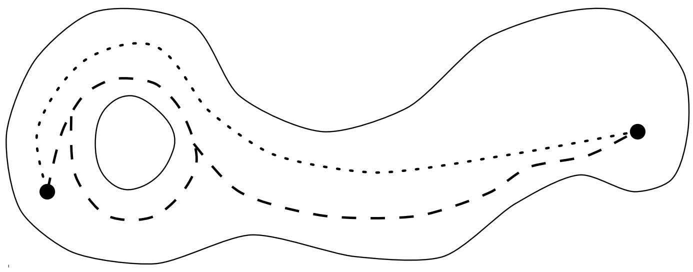

Figure 1: Different topological not equivalent paths

Def. 1 Given a two-dimensional digital *image* F with *pixels* $p \in \mathrm{X} \times \mathrm{Y} \times \mathrm{G}$, $\mathrm{X} = [0, n-1]$ the *columns*, $\mathrm{Y} = [0, n-1]$ the *rows* and $\mathrm{G} = [0, 255]$ or $\mathrm{G} = [-128, 127]$ the range of *grey values*

$$\begin{aligned} F : \mathrm{X} \times \mathrm{Y} &\longrightarrow \mathrm{G} \\ F(x, y) &\hookrightarrow [0, 255] \text{ or } [-128, 127] \end{aligned}$$

Def. 2 Two pixels $p_1 = (x_1, y_1, g_1)$ and $p_2 = (x_2, y_2, g_2)$ are called (8-)*neighbored* or (4-)*neighbored pixels*, respectively, if $| x_1 - x_2 | \leq 1, | y_1 - y_2 | \leq 1$ or $| x_1 - x_2 | + | y_1 - y_2 | < 2$, respectively.

Def. 3 A *path* is a sequence of neighbored pixels with a start pixel p_1 and an end pixel p_n. A set of pixels, where each pair of pixels is linked by a path completely contained in the set is a (path-)*connected set.*

Def. 4 A path is called *monotone increasing* or *decreasing path*, respectively, if the sequence of neighbored pixels from p_1 to p_n is monotone increasing or decreasing in terms of the grey values. This path is directed. A monotone increasing path from p_1 to p_n is a monotone decreasing path from p_n to p_1. It can be shown that this definition is an equivalence relation with a partition of the pixels into equivalence classes, the path connected components. In Fig.2 the literal understanding of paths is shown. Additionally possible borders of path connected components are sketched into.

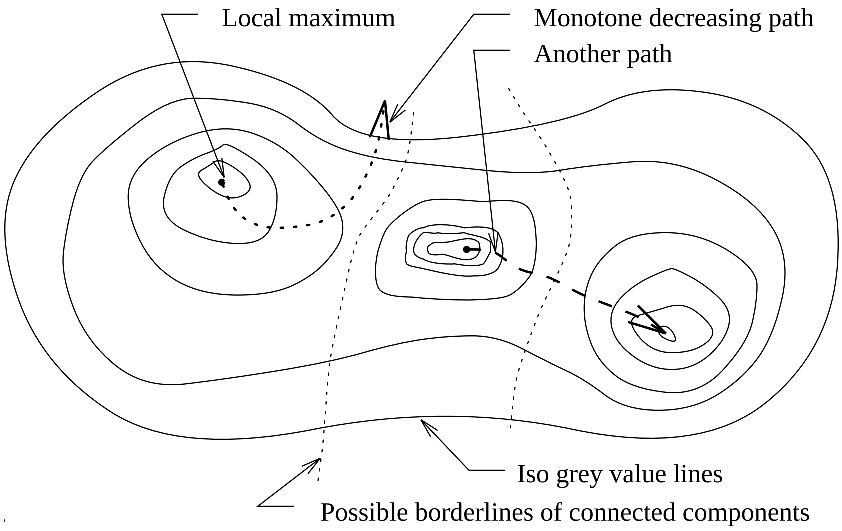

Figure 2: Illustration of paths

Def. 5 A *local maximum* (LMX) or *minimum* (LMN), respectively, is a connected set of pixels with constant maximum or minimum grey value in a certain surrounding.

Def. 6 Every local maximum or minimum, respectively, is related to *one upper* or *lower* (path-)*connected component* UCC or LCC. A connected component is a monotone increasing or decreasing path-connected set of pixels. We do not proof here that each set of connected components is a partition of the image. In Fig.3 a one-dimensional interpretation of these items is illustrated.

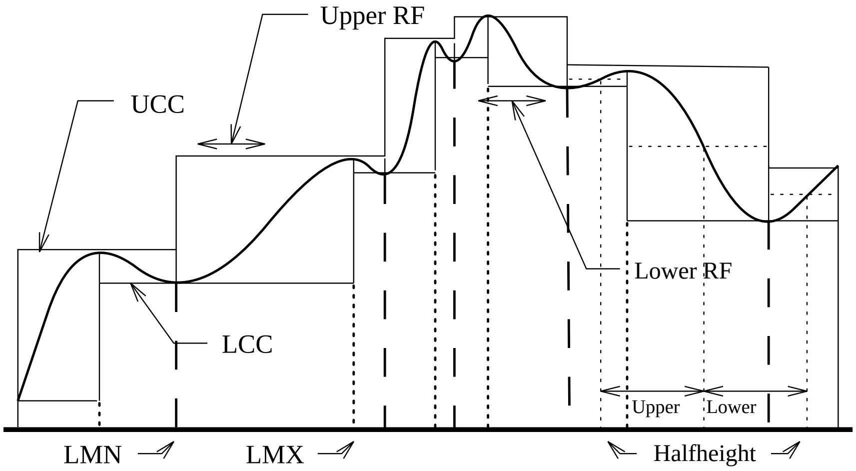

Figure 3: 1-dim illustration of ricefield transformation

Def. 7 From upper connected components a *(UCC-)neighborhood* relationship $\sim_1$ can be defined

$$UCC_i \sim_1 UCC_j \quad :\Leftrightarrow \quad \exists p_i \in UCC_i, \exists p_j \in UCC_j \text{with} p_i, p_j \text{neighbors}$$

The local maxima inherit the UCC-neighborhood canonically by $LMX_i \subset UCC_i$ and $LMX_j \subset UCC_j$. The *LCC-neighborhood* can be defined accordingly.

The combination of upper and lower connected components delivers a mixed neighborhood relation $\sim_2$ between both types of components in the following way: Let UCC_i be an upper connected component and $\mathbf{LCC}_i$ the *set* of lower connected components with non-empty intersection with UCC_i, than

$$UCC_i \sim_2 LCC \quad :\Leftrightarrow \quad LCC \in \mathbf{LCC}_i$$

defines a neighborhood between connected components as well as local maxima and (surrounding) local minima and *vice versa* (see Fig.4).

Def. 8 From the existence of monotone decreasing paths leading from one LMX to a $\sim_2$ neighbored LMN we can conclude the existence of exactly one point just on half height of the grey value range fom LMN to LMX. This is valid for every existing path, hence we can define all pixels in the upper half height of a path as the *upper half height set* UHH and the others as *lower half height set* LHH. This is illustrated in one dimension on the left of Fig.3. The calculation of the UHH and LHH is described with the results of the ricefield transformation in the next section. This partition is independent from absolute grey values and only dependent from the neighborhood of the local maxima and minima.

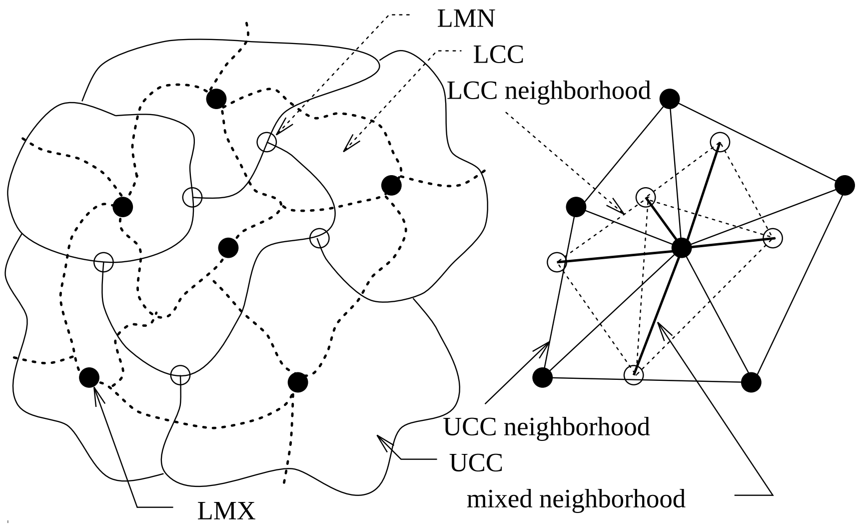

Figure 4: Neighborhoods from UCC and LCC, neighborhood graph

Image objects and their relations

Using terms as defined in the preceeding section, images can now be understood as sets of objects of the different types like UCC, LCC, LMX or LMN equipped with their neighborhoods. Depending from the actual application these merely mathematically defined terms have to be related to the actual meaning. The simplest example is an image representing a certain height map with the height coded as grey value. The UCC's represent the regions of single mountains limited by the path of lowest locations in the surrounding valleys and LCC's are the regions of the valleys surrounded by the watersheds and crest lines. LMX's represent the summits or peaks and the LMN's the holes.

Of special interest for picture evaluation are relations between objects. They allow the modelling of image content by relations of image objects. These relations can widely differ from local oriented neighborhoods up to those based on the grey value of the objects [6]. Relations are strongly related to the theory of graphs (see Fig.4).

As a first step for further description and structuring of an image the relations defined as neighborhoods in the preceeding section are needed. The relations represented as a graph can be examined and transformed with various methods. Again mathematical morphology is useful in this context [6, 12, 7].

Structuring and quantification of image objects

Image analysis considered as a hierachical description process delivers at every hierachical level quantitative features of increasing degree of abstraction. This procedure necessitates formal models. The methods described up to now are applicable to every hierachical level, however formulated for the objects of lowest level, the pixels.

Beside several pixel based quantitative features of the connected components like size or mean, their overall number can be used for calculation of a connectivity number. Comparable with the well known Euler number of sets, connectivity of the grey scale image can be defined as $E = \#UCC - \#LCC$.

As far as relations between objects exist well-known methods of graph analysis can be used for further analysis, see [14, 2]. At every level, new features can be defined on the base of all other lower-level features

Beside the topological tools described here, other methods, e.g. computational geometry, see [10], to define relations between objects can be used like the well-known Delaunay triangulation or heuristical methods [6]. E.g. the mixed neighborhood $\sim_2$ between local extrema can be understood as a star graph allowing various algorithmic methods to generate paths between objects or other groups of objects [2]. Defined graphs can either be merged or devided to define new objects with special properties.

Summary of topological applications

An image consists of pixels. Pixels are gathered in different connected components. The latter contain local extrema. Neighborhood relations allow the definition of entities like paths and connected components. New topologies are defined above objects for further more elaborated analysis of the image content on the level of newly defined objects.

Methods applied on digital images

Mathematical morphology

The body of methods in mathematical morphology is well known and described in [8, 9] and various other publications. We will only try to give a short impression about the algorithms used to generate the objects defined in the previous sections.

Frankly spoken the iterative application of the so called *hit-or-miss transformation* (HMT) results either in an *upper* or in a *lower ricefield image*. These images contain the connected components as well as the local extrema.

The HMT is a replacement rule which replaces a pixel by its neighbor, if the processed pixel is recognized as located between two pixels building a monotone increasing or decreasing path, respectively. The HMT is repeated as long as any replacement occurs. In fact it is a thickening or thinning algorithm.

A pixel configuration of an increasing or decreasing path is recognized by the application of directional oriented structuring elements containing one part for an erosion (estimation of minimum pixel) and a dilation (estimation of maximum pixel). The structuring elements are shown in Fig.5, filled circles stand for dilation and empty circels for erosion. The centre point of the structuring element is always considered as origin. The choice of the structuring elements guarantees the topological thickening or thinning, respectively. The central part of C-code used for the HMT is listed in Fig.6 [5]. The exact rule for or against a pixel replacement is given in [8].

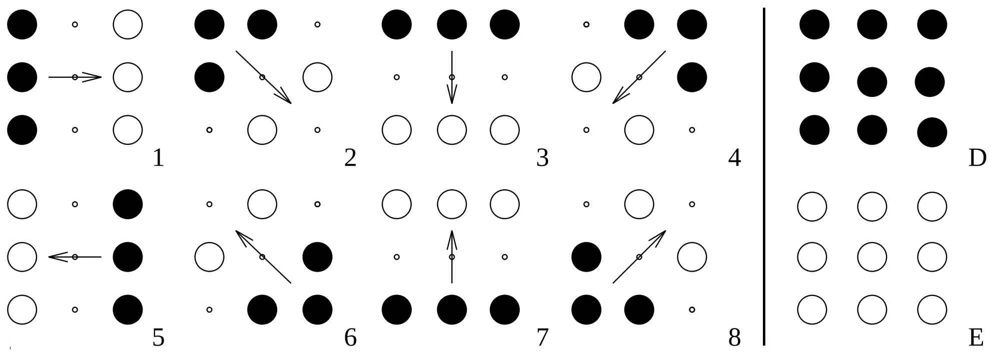

Figure 5: Sequence of structuring elements of the HMT

```
/* Hit-Or-Miss-Transformation with one Structuring Element              */
/* pmin, pmax arrays of pointers controlled by the structuring element   */
/* fmin, fmax functions returning minimum or maximum, respectively       */
/* n number of pixels in the image                                       */
/* ipo, ipi input and output image pointer                               */
  if (k && l) {
/* Thinning                                                              */
    if (vgl) for(i=0;i<n;i++) {
      min = fmin(pmin), /*Erosion part*/ max = fmax(pmax);/*Dilation part*/
      if ((max < *ipo) && (min >= *ipo)) *ipi = max; else *ipi = *ipo;
      ipi++, ipo++; }
/* Thickening                                                            */
    else  for(i=0;i<n;i++) {
      min = fmin(pmin), max = fmax(pmax);
      if ((max <= *ipo) && (min > *ipo)) *ipi = min; else *ipi = *ipo;
      ipi++, ipo++; }
  } else {
/* Dilation                                                              */
    if (k) for (i=0;i<n;i++) *ipi++ = fmax(pmax),ipo++;
/* Erosion                                                               */
    else if (l) for (i=0;i<n;i++) *ipi++ = fmin(pmin),ipo++; }
```

Figure 6: Part of C-code of the HMT

Using the familiar notations of mathematical morphology a structuring element

$T_i = (T_i^\bullet, T_i^\circ)$ is applied and compared with the original pixel grey value by

$$HMT(F,T_i)(x,y) = \begin{cases} F \oplus T_i^\bullet(x,y) & : \text{if } F \oplus T_i^\bullet(x,y) < F(x,y) \leq F \ominus T_i^\circ(x,y) \\ F(x,y) & : \text{else} \end{cases}$$

and is performed for each directed structuring element (see Fig.5) as long as any replacement occurs. This formula delivers a thinning transformation. The formula for thickening is analogous

$$HMT(F,T_i)(x,y) = \begin{cases} F \ominus T_i^\circ(x,y) & : \text{if } F \oplus T_i^\bullet(x,y) \leq F(x,y) < F \ominus T_i^\circ(x,y) \\ F(x,y) & : \text{else} \end{cases}$$

(see Fig.6). The number of necessary iterations depends from the image topology and is directly related to the longest path in the image. The resulting image is called ricefield since it has some similarity with ricefield terraces. The 1$^{\text{st}}$ formula given leads to the *lower ricefield* (LF). The 2$^{\text{nd}}$ formula results in the *upper ricefield* (UF).

Results of ricefield transformations

The resulting image consists of flat connected components (UCC/LCC) with next existing grey value just below or above the local maximum or minimum (LMX/LMR), respectively. The grey values of the deviding lines of the connected components are those of the original image.

In spite of the theoretical definition of connected components the algorithm cannot generate connected components touching each other. Thin (4-neighbored) lines (paths) lie in between. This results from the complicated topology of discrete sets without open sets and without borders of area 0. In Figure 7b the digital image of a cell nucleus, an osteoblast, is shown with the upper and lower ricefield transform in the first image row (Fig.7a,7c). The dark lines in the left (UR) and the bright lines in the right (LR) image are the deviding lines mentioned above. In the second row of Figure 7 the connected components (Fig.7d,7f). In the Figure 8 the $\sim_1$ neighborhoods are shown (Fig.8a,8c) as well as the $\sim_2$ neighborhood (Fig.8b). The nodes of the graphs are located at the centroids of the LMN's and LMX's. The neighborhood graphs are shown on the complete nucleus to avoid border effects possible in the enlarged image.

For the estimation of upper and lower half height sets (UHH/LHH) the deviding lines of the connected components in UF and LF are desturbing. Considering a half height image calculated with

$$HH = \frac{UF + LF}{2} - F$$

(see Fig.3) results in an image with grey values in $[-128, 127]$. This image has theoretically pixel grey values ≥ 0 for pixels in the *upper half height* and pixel grey values < 0 in the *lower half height* (see Fig.3). Unfortunately the borders of the connected components (thin lines as already mentioned) have original image values, hence they have

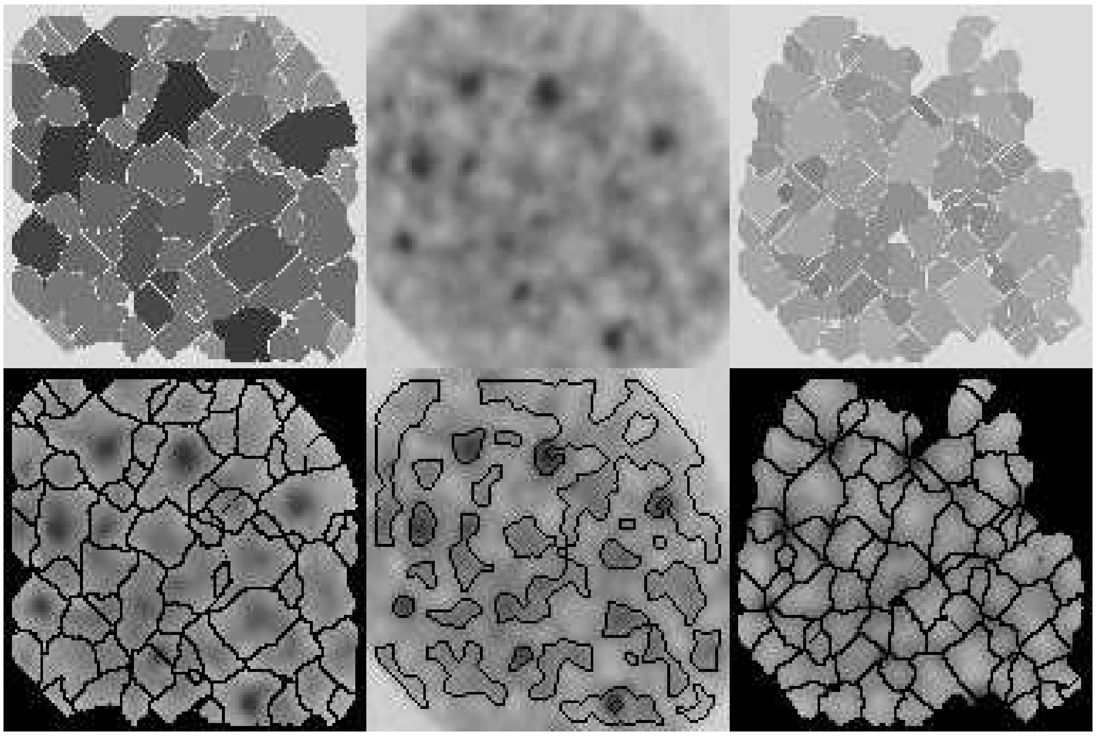

Figure 7: $\begin{array}{c|c|c} \text{a} & \text{b} & \text{c} \\ \hline \text{d} & \text{e} & \text{f} \end{array}$ Ricefields and components (see text)

in the HH image grey value 0 according to the given formula. To avoid these unwished pixels, we apply the following formula for calculation of the half height image:

$$HH = \frac{UF^{T^{\bullet}_{D}} + LF_{T^{\circ}_{E}}}{2} - F$$

In Fig.9a and Fig.9c the closing UF^{T_D} and the opening LF_{T_D} (see the structuring element D in Fig.5) of the ricefield images UF and LR are shown. The closing is a dilation succeeded by an erosion and the opening is an erosion succeeded by a dilation. The program code used is shown in Figure 6.

In Fig.7e the upper half height set (UHH) is shown. One can recognize, that the UHH appears smaller as the LHH, which is the complement of the UHH. This difference is true. The profile of the grey values in this example image of a cell nucleus has higher peaks (dark grey values) and smoother valleys, or with other words, the local extrema are more distant from each other. Hence the half height line is nearer to the LMX and accordingly the area of the UHH is smaller than the one of the LHH.

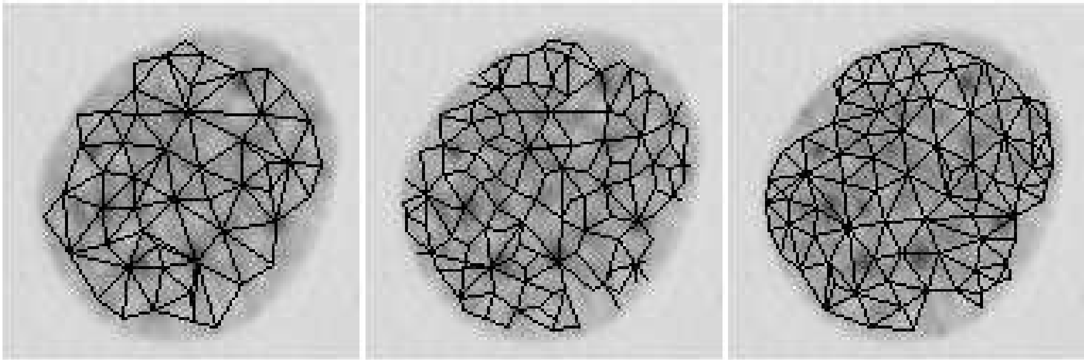

Figure 8: a | b | c Neighborhood graphs UCC, mixed and LCC (see text)

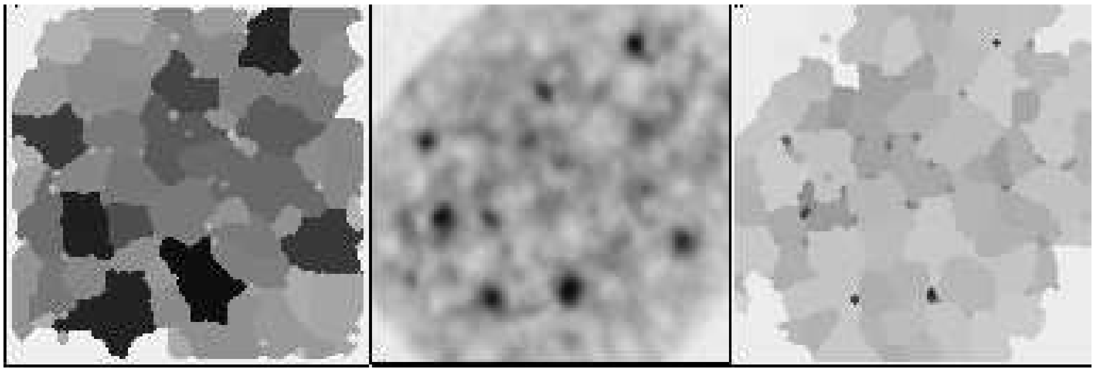

Figure 9: a | b | c Closing of UF and opening of LF

Summary of methods applied

The application of the hit-or-miss-transformation for thickening (upper ricefield) and thinning (lower ricefield) delivers transformed images with several interesting and useful properties. The upper ricefield image contain the upper connected components and local maxima, the lower ricefield the lower connected components and the local minima. The ricefields can be interpreted as an upper and lower *hull* of the image (see Fig.3) and hence additionally useful for calculation of a half height partition, derived from the difference between the mean value image of upper and lower ricefield and the original. This procedure is comparable with segmentation methods applied in linear system theory, using a band pass filtered image instead, taking the difference with the original and the application of a threshold.

The computing time of the ricefield transformation is direct dependent from the longest free path inside the image, the number of elements of this path. To give an impression of the transformation the UR and LR is shown in Fig.10 in pseudo color, illustrating not only the connected components but also the included local extrema.

The ricefield transformation is sensitive for the slightest changes in grey values.

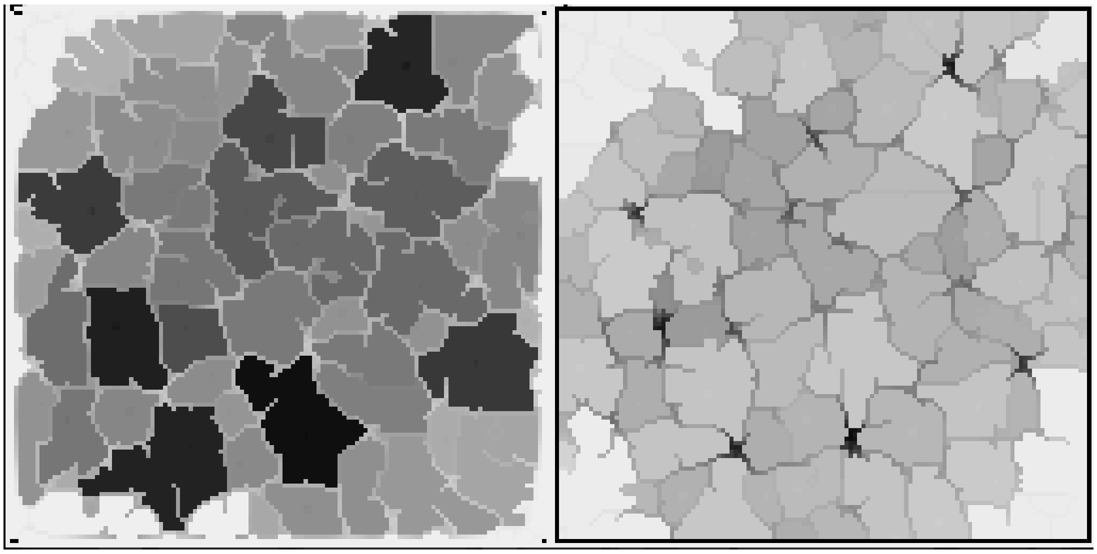

Figure 10: Upper and lower ricefield

Some noise will break down connected components to small pieces. *Vice versa* even strong averaging do *not* change the long range results, hence gives not false results. A smoothing with an appropriate filter size will avoid connected components from the size of the filter used and of smaller size.

Summary

Using topological entities like neighborhood, path, connected path and monotonicity, concepts are outlined for a structural description of image content without a-priori.

Image objects are defined as path connected sets of pixels, one group by increasing paths and the other by decreasing paths. Both groups build a partition of the image up to 4-connected thin deviding lines. A local neighborhood of the objects of each group can be defined by common deviding lines. The unique local extremum of each object can be considered as support of the image object supplied with the appropriate neighborhood and quantitative features (of the image object).

The algorithms are presented for digital images above square grids and illustrated for the densitometrical image of a cell nucleus. There is no restriction applying the methods to any other image or transformation of an image.

The resulting image description consisting of the objects, their neighborhoods and features can be considered as a weighted or marked graph accessible for further steps of analysis. In the laboratory of the author these graphs are extensively used for the quantitative description of local interdependencies in tissue architecture and chromatin expression.

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